Branched covers + lecture 6: CG invts as sliceness obst

Note correction from last time

$$Bl : \mathcal{A}(K) \times \mathcal{A}(K) \longrightarrow \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$$

$$x \qquad y \longmapsto \frac{1}{q(t)} \sum_{-\infty}^{\infty} (D \cdot t^i y)\, t^i \quad \text{not } t^{-i}$$

$\exists\, q(t) \neq 0$ s.t. $q(t)x = 0$

$\Rightarrow \exists D$ s.t. $\partial D = q(t)x$

Other announcements:

- May 15 help session: by me, not Hyeonhee (out of town) (May 22 by Isaac)
- Course evals next week (including for online/non-student participants) [mention anonymous feedback on website]

Last time: CG signature of a 3-mfld Y + $\chi : \pi_1 Y \to \mathbb{Z}/m$, $m \geq 1$

Goal: get sliceness obstr. for a knot.

Plan: associate a 3-mfld to a knot.

Branched covers

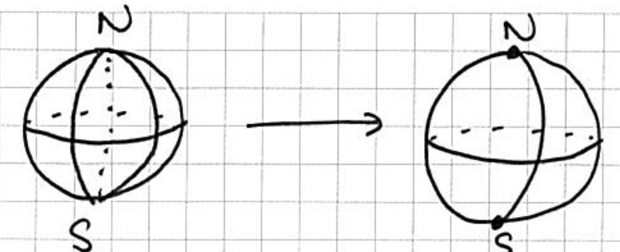
A d-fold (cyclic) branched cover $f : X^n \to Y^n$ with branching set $B \subseteq Y$ is a continuous map s.t.

(i) $f|_{X \setminus f^{-1}(B)} : X \setminus f^{-1}(B) \to Y \setminus B$ is a d-fold (cyclic) covering map

(ii) $\forall p \in f^{-1}(B)$ $\exists$ charts $\mathcal{U}, V \to \mathbb{C} \times \mathbb{R}^{n-2}$ (if $p \in \partial X$, use $\mathbb{R}^{n-2}_+$) about p, $f(p)$ on which f is given by $(z, x) \longmapsto (z^m, x)$ for some $m \geq 1$ called the branching index at p.

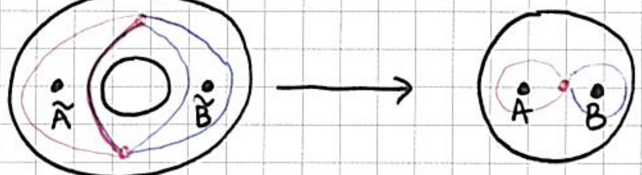
Note: branching index might differ at the elements of $f^{-1}(B)$. [but not for us] stick to cyclic

E.g. $\mathbb{C} \longrightarrow \mathbb{C}$, $z \longmapsto z^m$ — m-fold "cyclic" branched cover, branched along $0 \in \mathbb{C}$.

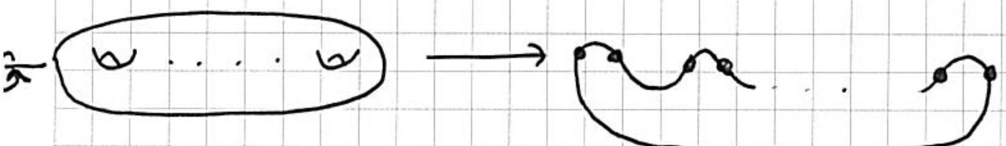


$(z,t) \longmapsto (z^m, t)$

m-fold branched cover, branched along $\{N,S\}$



Build a 2 fold cover branched along A, B, s.t. both have branching index 2.

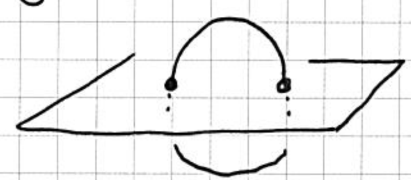


Σ_g $\qquad$ S^2

2-fold branched cover with $2g+2$ branch pts.

What about $n=3$?

E.g. S^3 branched along unknot



$S^3 = B^3 \cup B^3$

$\quad\;\; \cup\!\!\!\mid \qquad \cup\!\!\!\mid$

$\mathcal{U} = \text{arc} \cup \text{arc}$

B^3, arc unknotted $\xleftarrow{m\text{ fold}} B^3$

B^3, arc unknotted $\xleftarrow{m\text{-fold}} B^3$

note these are $(B^2, pt) \times I \leftarrow (B^2) \times I$

Glue together $\rightsquigarrow S^3$ again.

Alternatively: $S^3 \setminus \mathring{\nu}\mathcal{U} \cong S^1 \times D^2$

$m\mu_u \uparrow$ $\quad \uparrow m$-fold cover

$S^1 \times * \quad S^1 \times D^2$

Glue in solid torus.

Result: S^3 again.

In general, for cyclic covers of S^3, branched along $K \subseteq S^3$,

use $\pi_1(S^3 \setminus \mathring{\nu} K) \twoheadrightarrow \mathbb{Z}/m$ to get unbranched cover of the complement $\widehat{S^3 \setminus \mathring{\nu} K}$.

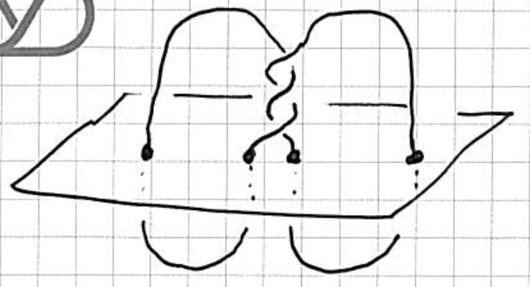
Note $[\mu_K] \longmapsto 1$.

so $m[\mu_K]$ lifts to a loop.

Then $\Sigma_n(K) = \widehat{S^3 \setminus \mathring{\nu} K} \cup S^1 \times D^2$

$\widehat{m[\mu_K]} \longleftrightarrow \partial D^2$

E.g. Σ_2(Trefoil)



B^3, 2 arcs $\xleftarrow{\text{unknotted 2 fold}}$ $S^1 \times D^2$

B^3, 2 arcs $\xleftarrow[\text{2 fold}]{\text{trivial}}$ $S^1 \times D^2$

Glue together $\rightsquigarrow$ lens space!

in fact, $L(3,1) = \mathbb{RP}^3$.

A Casson-Gordon sliceness obstruction [special case]

$K \subseteq S^3$ slice knot.

1. Then $|H_1(\Sigma_2(K);\mathbb{Z})|$ is a square, say m^2
2. If $\Sigma_2(K)$ is a lens space and $\chi: \pi_1(\Sigma_2(K)) \to \mathbb{Z}/\ell$ is a nontrivial map with ℓ a prime power with $\ell | m$. then $|\sigma_{CG}(\Sigma_2(K), \chi)| \leq 1$.

[sketch of proof in the end?]

[More generally, $K \subseteq S^3$ slice, n prime power. $\exists P \leq H_1(\Sigma_n(K))$ metaboliser wrt lk form s.t. $\forall$ prime power m $\forall \chi: H_1(\Sigma_n(K)) \to \mathbb{Z}/m$ with $\chi|_P = 0$ have $|\sigma_{CG}(\Sigma_n(K), \chi)| \leq \dim H_1^{\chi}(\Sigma_n(K), \mathbb{C}) + 1$

where we get $H_1^{\chi}(\Sigma_n(K);\mathbb{C})$ by thinking of χ mapping to $\mathbb{C}$.]

[metaboliser: $lk|_{P\times P} = 0$ and $|P|^2 = |H_1(\Sigma_n(K))|$].

Computational tool [Casson-Gordon] + long example.

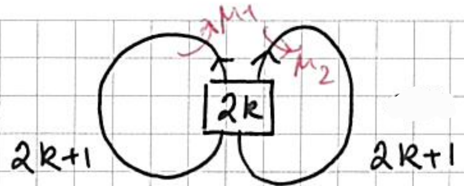
Consider: $T_k :=$  twist knots.

HW: T_k is algebraically slice iff $k = 0, 2, 6, \dots$

$\left.\begin{array}{l} T_0 = \text{unknot} \\ T_2 = \text{stevedore} \end{array}\right\}$ are slice (in fact ribbon)

[Casson-Gordon] T_k slice $\Leftrightarrow$ $k = 0, 2$.

HW : $\Sigma_2(T_k)$ given by



HW: this is a lens space! in fact $L(4k+1, 2)$

Goal: $|\sigma_{CG}(\Sigma_2(T_k), \chi)| > 1$ for $k = 6$.

need to find this.

HW: $H_1(\Sigma_2(T_k))$ has pres. matrix $\Lambda_k = \begin{array}{cc} \mu_1 & \mu_2 \end{array}$ $\begin{bmatrix} 2k+1 & 2k \\ 2k & 2k+1 \end{bmatrix}$

Note: $|H_1(\Sigma_2(T_k))| = 4k+1 = m^2$ for some m.

HW (part of alg slice)

"linking - framing matrix"

Sketch of proof: $H_1(Y \setminus L)$ gen. by meridians
relations from the surgery coeffs

How to compute σ_{CG}?

[Casson-Gordon] Y^3 closed, oriented, obtained by integral Dehn surgery on an n-comp link $L = \{L_i\}_{i=1}^n$ with linking-framing matrix $\Lambda = (a_{ij})$.

Suppose $\chi : H_1(Y) \twoheadrightarrow \mathbb{Z}/p$, where p is a prime power & $\chi(\mu_{L_i}) = 1 \quad \forall i$.

<u>oriented</u> meridian

Then $\sigma_{CG}(Y, \chi) = \sigma_{e^{2\pi i/p}}(L) - \sigma(\Lambda) + 2\dfrac{(p-1)}{p^2} \sum_{i,j=1}^{n} a_{ij}$.

Levine-Tristram signature of L at $e^{2\pi i/p}$

Can we define such a χ for $\Sigma_2(T_k)$?

Set $\chi(\mu_1) = 1 = \chi(\mu_2) \in \mathbb{Z}/p$ for some $p | m$

To extend to $H_1(\Sigma_2(T_k))$, check relations: $(2k+1)\mu_1 + 2k\,\mu_2 = 0$
$(2k)\mu_1 + (2k+1)\mu_2 = 0$

$\Big\}\chi$

$4k+1 \overset{?}{=} 0$

Yes since $p | m^2 = 4k+1$

What about $\sigma_{e^{2\pi i/p}}(L)$

torus link $T_{2,4k}$

(2,4k)-torus link

unfortunate notation

[Litherland] computed σ_ω for torus links!

$k=2$, $m^2 = 4k+1 = 9$, $p=3$, $\sigma_{2\pi i/3}((2,8)\text{-torus link}) = -5$

$k=6$, $m^2 = 4k+1 = 25$, $p=5$, $\sigma_{2\pi i/5}((2,24)\text{-torus link}) = -9$.

$$\sigma_{CG}(\Sigma_2(T_2),\chi) = \sigma_{2\pi i/3}(L) - \sigma(\Lambda_2) + 2\,\frac{(3-1)}{3^2}\sum a_{ij}$$
$$= -5 - 2 + 8$$
$$= 1.$$ no obstruction to slice

$$\sigma_{CG}(\Sigma_2(T_6),\chi) = \sigma_{2\pi i/5}(L) - \sigma(\Lambda_6) + 2\,\frac{(5-1)}{5^2}\sum a_{ij}$$
$$= -9 - 2 + 16$$
$$= 5$$ not slice! Therefore $\exists$ alg slice, non slice knots

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<u>More direct Casson-Gordon invt of knots</u> [Didn't cover in lecture]

$K \subseteq S^3 \rightsquigarrow M_K := S^3_0(K)$ 0-framed Dehn surgery.

$M_n(K) :=$ n-fold cyclic cover of M_K [note $H_1(M_K) \cong \mathbb{Z}$]

$$\boxed{\text{Input: } \chi : H_1(\Sigma_n(K)) \longrightarrow \mathbb{Z}/m \qquad m \geq 1}$$

Also $\alpha : \pi_1(M_n(K)) \longrightarrow \pi_1(M_K) \longrightarrow H_1(M_K) \cong \mathbb{Z}$

Then
$$\begin{tikzcd}
\pi_1(M_n(K)) \arrow[r, "\alpha\times\chi"] \arrow[d] & \mathbb{Z}\times\mathbb{Z}/m \\
H_1(M_n(K)) \arrow[d, "\cong"', "HW"] & \\
H_1(\Sigma_n(K)) \oplus \mathbb{Z} \arrow[uur, "\chi"] \arrow[uur, "0"'] &
\end{tikzcd}$$

As before, $\exists\, r \geq 1$ s.t. $r(M_n(K), \alpha\times\chi) = \partial(V_n, \varphi)$

where V_n compact, oriented,
$$\begin{tikzcd}
\pi_1(V_n) \arrow[r, "\varphi"] & \mathbb{Z}\times\mathbb{Z}/m \arrow[r] & \mathbb{C}(t) \\
\pi_1(M_n(K)) \arrow[u, "inc_*"] \arrow[ur, "\alpha\times\chi"'] & &
\end{tikzcd}$$
$(1 \mapsto t,\ 1 \mapsto e^{2\pi i/m})$

$\rightsquigarrow H_2^{\varphi}(V_n;\mathbb{C}(t))$ with twisted int form $Q^{\varphi}_{V_n,\mathbb{C}(t)} \in W(\mathbb{C}(t))$

Hermitian + if m prime power, nonsing

Witt group

Similarly have $Q_{V_n,\mathbb{Q}}$ on $H_2(V_n;\mathbb{Q})$

[might be singular]

but get non-singular int form on $H_2(V_n;\mathbb{Q}) / \mathrm{im}\left(H_2(\partial V) \to H_2(V)\right)$

Call this $Q^{\text{non-sing}}_{V_n,\mathbb{Q}} \in W(\mathbb{Q})$

$$\begin{array}{c} W(\mathbb{Q}) \\ \downarrow i \\ W(\mathbb{C}(t)) \end{array}$$

$$\tau(K,\chi) := \left(\left[Q^{\varphi}_{V_n,\mathbb{C}(t)}\right] - i\left[Q^{\text{non-sing}}_{V_n,\mathbb{Q}}\right]\right) \otimes \frac{1}{r} \in W(\mathbb{C}(t)) \otimes \mathbb{Q}$$

<u>Another sliceness obstruction</u> [Casson-Gordon]

$K \subseteq S^3$ slice, q a prime power.

$\exists G \leqslant H_1(\Sigma_q(K))$ s.t. (i) $|G|^2 = |H_1(\Sigma_q(K))|$

(ii) $lk_{\Sigma_q(K)}\big|_{G\times G} = 0$

"metabolizer" for lk

(iii) if $\chi : H_1(\Sigma_q(K)) \to \mathbb{Z}/m$, m prime power

satisfies $\chi(G) = 0$

then $\tau(K,\chi) = 0$.